① Solve $y(x) = f(x) + \lambda \int_0^1 xt \, y(t)\, dt$

<u>Sol:-</u> The given eqⁿ is

$$y(x) = f(x) + \lambda \int_0^1 xt \, y(t)\, dt \quad \text{—— ①}$$

Compare with

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\, y(t)\, dt$$

Here $f(x) = f(x)$, $K(x,t) = xt$
$= a_1(x)\, b_1(t)$

where $a_1(x) = x$, $b_1(t) = t$

Solⁿ of eqⁿ ① is

$$y(x) = f(x) + \lambda [c_1 a_1(x)]$$

$$y(x) = f(x) + \lambda [c_1(x)] \quad \text{—— ②}$$

$$a_{11} = \int_0^1 b_1(t)\, a_1(t)\, dt = \int_0^1 t \cdot t \, dt = \int_0^1 t^2\, dt = \left[\frac{t^3}{3}\right]_0^1$$

$$= \frac{1}{3}[1-0]$$

$$= \frac{1}{3}$$

$$a_{12} = \int_0^1 b_1(t)\, a_2(t)\, dt = 0$$

$$f_1 = \int_0^1 b_1(t)\, f(t)\, dt = \int_0^1 t \cdot f(t)\, dt$$

<u>To find c_1:-</u>

$$(1-\lambda a_{11})c_1 - \lambda a_{12} c_2 = f_1$$

$$\left(1-\lambda\cdot\frac{1}{3}\right)c_1 - \lambda(0)c_2 = \int_0^1 t\cdot f(t)\,dt$$

$$\left(\frac{3-\lambda}{3}\right)c_1 = \int_0^1 t\cdot f(t)\,dt$$

$$c_1 = \frac{3}{3-\lambda}\int_0^1 t\, f(t)\,dt \quad \text{where } \lambda \neq 3$$

Put the value of c_1 in eq^n (2), we get

$$y(x) = f(x) + \lambda\left[\frac{3}{3-\lambda}\int_0^1 t\cdot f(t)\,dt\right]x$$

$$= f(x) + \frac{3\lambda x}{3-\lambda}\int_0^1 t\cdot f(t)\,dt$$

Which is the required sol^n where $\lambda \neq 3$.

(2) Solve $y(x) = \cos x + \lambda \int_0^{\pi} \sin(x+t)\, y(t)\, dt$

Solⁿ :- The given eqⁿ is

$$y(x) = \cos x + \lambda \int_0^{\pi} \sin(x+t)\, y(t)\, dt$$

$$y(x) = \cos x + \lambda \int_0^{\pi} (\sin x \cos t + \cos x \sin t)\, y(t)\, dt \quad \text{— (1)}$$

Compare with

$$y(x) = f(x) + \lambda \int_a^b k(x,t)\, y(t)\, dt$$

where $f(x) = \cos x$, $k(x,t) = \sin x \cos t + \cos x \sin t$
$= a_1(x)\, b_1(t) + a_2(x)\, b_2(t)$

$a_1(x) = \sin x$	$a_2(x) = \cos x$
$b_1(t) = \cos t$	$b_2(t) = \sin t$

∴ Solⁿ of eqⁿ (1) is

$$y(x) = \cos x + \lambda [a_1 c_1 + a_2 c_2]$$
$$= \cos x + \lambda [\sin x\, c_1 + \cos x\, c_2] \quad \text{— (2)}$$

$$a_{11} = \int_0^{\pi} b_1(t)\, a_1(t)\, dt = \int_0^{\pi} \cos t \cdot \sin t\, dt$$

$$= \frac{1}{2} \int_0^{\pi} 2 \cos t \sin t\, dt$$

$$= \frac{1}{2} \int_0^{\pi} \sin 2t\, dt = 0$$

$$a_{12} = \int_0^{\pi} b_1(t)a_2(t)dt = \int_0^{\pi} \cos t \cdot \cos t \, dt$$

$$= \int_0^{\pi} \cos^2 t \, dt$$

$$= \int_0^{\pi} \left(\frac{\cos 2t + 1}{2} \right) dt$$

$$= \frac{1}{2} \left[\frac{\sin 2t}{2} + t \right]_0^{\pi}$$

$$= \frac{1}{2} \left[\frac{\sin 2\pi}{2} + \pi - \left(\frac{\sin 2(0)}{2} + 0 \right) \right]$$

$$= \frac{1}{2} \left[0 + \pi - 0 \right]$$

$$= \pi/2$$

$$a_{21} = \int_0^{\pi} b_2(t)a_1(t)dt = \int_0^{\pi} \sin t \cdot \sin t \, dt$$

$$= \int_0^{\pi} \sin^2 t \, dt$$

$$= \int_0^{\pi} \left(\frac{1 - \cos 2t}{2} \right) dt$$

$$= \frac{1}{2} \left[t - \frac{\sin 2t}{2} \right]_0^{\pi}$$

$$= \frac{1}{2} \left[\pi - \frac{\sin 2\pi}{2} - \left(0 - \frac{\sin 2(0)}{2} \right) \right]$$

$$= \pi/2$$

$$a_{22} = \int_0^{\pi} b_2(t)\, a_2(t)\, dt = \int_0^{\pi} \sin t \cdot \cos t\, dt$$

$$= \frac{1}{2}\int_0^{\pi} 2 \sin t \cos t\, dt$$

$$= 0$$

$$f_1 = \int_0^{\pi} b_1(t)\, f(t)\, dt = \int_0^{\pi} \cos t \cdot \cos t\, dt$$

$$= \int_0^{\pi} \cos^2 t\, dt$$

$$= \pi/2$$

$$f_2 = \int_0^{\pi} b_2(t)\, f(t)\, dt = \int_0^{\pi} \sin t \cdot \cos t\, dt = \frac{1}{2}\int_0^{\pi} 2 \sin t \cos t\, dt$$

$$= 0$$

<u>To find c_1 and c_2 :-</u>

$$(1-\lambda a_{11}) c_1 - \lambda a_{12} c_2 = f_1$$

$$-\lambda a_{21} c_1 + (1-\lambda a_{22}) c_2 = f_2$$

i.e $$(1-\lambda(0)) c_1 - \lambda \pi/2\, c_2 = \pi/2$$

$$-\lambda \pi/2\, c_1 + (1-\lambda(0)) c_2 = 0$$

i.e $$\left.\begin{aligned} c_1 - \lambda \tfrac{\pi}{2} c_2 &= \pi/2 \\ -\lambda \pi/2\, c_1 + c_2 &= 0 \end{aligned}\right\} \quad \text{--- (3)}$$

For Infinite or no sol. $D(\lambda) = 0$

~~The system (3) has sol^n~~

$$D(\lambda) = 0$$

$$\Rightarrow \begin{vmatrix} 1 & -\lambda \pi/2 \\ -\lambda \pi/2 & 1 \end{vmatrix} = 0$$

$$1 - \frac{\lambda^2 \pi^2}{4} = 0$$

$$\Rightarrow 1 = \frac{\lambda^2 \pi^2}{4}$$

$$4 = \lambda^2 \pi^2 \Rightarrow \lambda^2 \pi^2 = 4$$

$$\lambda^2 = \frac{4}{\pi^2} \Rightarrow \boxed{\lambda = 2/\pi}$$

Put the value of $\lambda = 2/\pi$ in eq^n (3)

$$c_1 - \frac{\cancel{2}}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{\cancel{2}} c_2 = \pi/2$$

$$-\frac{\cancel{2}}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{\cancel{2}} c_1 + c_2 = 0$$

$$\Rightarrow \quad c_1 - c_2 = \pi/2$$

$$-c_1 + c_2 = 0$$

$$\Rightarrow \quad c_1 - c_2 = \pi/2 \quad \text{—— (4)}$$

$$c_1 - c_2 = 0 \quad \text{—— (5)}$$

The system of eq^n (4) & (5) is Inconsistent and so it possesses no sol^n
Hence c_1 and c_2 can't be determined. It shows that given Integral eq^n possesses no sol^n.